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OPTIMAL SCHEDULING IN SOME MULTI-QUEUE SINGLE-SERVER SYSTEMS

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Ordonnancement Optimal dans Certains Systèmes de Files d'Attente en Parallèle et Serveur Unique

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Résumé

L'allocation optimale d'un serveur unique à plusieurs files d'attente en parallèle est étudiée dans cet article. La durée d'une visite du serveur à une file particulière est aléatoire et ne dépend pas de l'état de la file visitée. Deux variantes sont considérées: la file visitée est vidée à la fin de chaque visite (variante I) ou seuls les clients présents à l'arrivée du serveur quittent le système à la fin de sa visite (variante II). La politique d'ordonnancement détermine la prochaine file d'attente que devra visiter le serveur. Sous l'hypothèse où les processus des arrivées sont homogènes, les résultats suivants sont obtenus: dans le cas où l'ordonnanceur ne connaît pas l'état du système, nous montrons qu'une politique *cyclique* minimise, à chaque instant, le nombre moyen de clients dans le système. Dans le cas où l'ordonnanceur connaît l'état de chaque file d'attente, nous montrons que la politique qui alloue le serveur à la file la plus longue minimise, à chaque instant et pour l'ordre stochastique fort, un vecteur ordonné du nombre de clients. Ces résultats sont vrais pour les variantes I et II et sont établis sous des hypothèses statistiques très générales. Ce modèle s'applique à certains protocoles de communication multi-accès (e.g. TDMA) ainsi qu'au système *videotex*.

Mots-Clès: Files d'Attente en Parallèle; Ordonnancement Optimal; Ordre Stochastique; Couplage; Service Cyclique; Protocoles Multi-Accès; TDMA; Système *Videotex*.

Optimal Scheduling in Some Multi-Queue Single-Server Systems

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Abstract

In this paper we address the problem of optimal scheduling in a Multi-Queue Single-Server (MQSS) model. The server visits N queues in an arbitrary manner. Each queue is visited for a random period of time whose duration is sampled in advance. At the end of a visit period, either all customers of the attended queue leave the system (variant I), or only customers that were present in the queue upon the arrival of the server leave the system (variant II). A scheduling policy is a rule that selects the next queue to be visited by the server. When the controller has no information on the state of the system, it is shown, under homogeneous arrival assumptions, that a *cyclic policy* minimizes the expected number of customers in the system. When the controller knows the number of customers in each queue, it is shown that the so-called *Most Customers First* (MCF) policy minimizes, in the sense of strong stochastic ordering, the vector of the number of customers in each queue whose components are arranged in decreasing order. These results hold for variants I and II, and are obtained under fairly weak statistical assumptions. This model has potential applications in videotex and time division multiple access systems.

Keywords: Multi-Queue Single-Server; Optimal Scheduling; Stochastic Ordering; Coupling; Interchange Arguments; Cyclic Service; Most Customers First Policy; Broadcast Service; Time Division Multiple Access; Videotex System.

1 Introduction

Consider the following Multi-Queue Single-Server (MQSS) model. There are $N \geq 1$ queues and a single server. An arriving customer enters queue j with the probability $q_j > 0$, $q_1 + \cdots + q_N = 1$. The server visits (serves) each queue in an arbitrary manner for a random period of time whose duration is sampled in advance. Once a service has started, the server cannot leave the queue before the visit time has expired (*nonpreemptive* service). At the end of a visit period, either all customers in the visited queue leave the system (variant I), or only customers that were present in the queue upon the arrival of the server leave the system (variant II).

Our objective is to determine optimal scheduling policies. A *scheduling policy* is a rule that the controller uses for selecting the next queue to be visited by the server. Two different cases will be investigated, depending upon the amount of information available to the controller.

In the first case, the controller has no information on the state of the system (i.e. number of customers in each queue). In particular, this implies that the server may visit an empty queue even if the system is nonempty. This model will be referred to as the Decentralized MQSS (D-MQSS) system.

In the second case, the controller knows the number of customers in each queue at the decision epochs. Here, the server is allocated to a queue *if and only if* that queue is nonempty. This implies that only *nonidling* or *work conserving* policies will be considered. This model will be referred to as the Centralized MQSS (C-MQSS) system.

The MQSS model arises in some communication networks. An example is the Time Division Multiple Access (TDMA) system where several stations share a single communication channel [6]. When the channel is allocated to a station, all the messages contained in this station are transmitted (i.e., variant I). Within this context, the queues model the stations, and the server represents the communication channel. In this system, the scheduling policy decides which station must be given the right of transmission, or, equivalently stated, which station the channel is to be allocated to at the beginning of each transmission period. In the case where the channel is slotted and each station has a buffer of length one (a message is lost if the buffer is occupied upon its arrival), Itai and Rosberg showed [5] that an optimal policy that maximizes the system throughput can be found in the class of *cyclic policies*, provided the arrivals to each station in a slot form Bernoulli independent random variables, and that the system is of decentralized type.

Another example arises in the *videotex* system, which is a retrieval and delivery system with broadcast service (see [1], [3]). In such a system, there are, say, N pages of information and there is a computer system maintaining the retrieval and the delivery service of those pages. Various user terminals are connected to the service computer through a broadcast network. The computer system delivers the pages by broadcasting information to all the user terminals via the network. When a user requires a page, the user terminal senses the communication medium until the desired page is detected. The page is then captured, stored and displayed. Here, again, the system can be modeled by our MQSS model, where the queues represent the information pages, the server represents the service computer and the customers are the page requests sent by the user terminals. Scheduling in such a model consists of deciding which page is to be retrieved and delivered at the beginning of each transmission period. Under the assumptions that the system is slotted (i.e., the time to retrieve and deliver any page is exactly one slot) and that within a slot customers arrive at queue j according to a Poisson process with intensity $\lambda_j > 0$, Ammar and Wong [1] showed that a cyclic policy minimizes the mean response time, provided the system is of decentralized type and that a request for page j arriving at the computer system during a page j transmission has to wait until the next transmission of page j, $j = 1, 2, \ldots, N$ (i.e., the D-MQSS model with variant II).

For the variant I of the centralized videotex system with equal page requirement probabilities ($q_1 = \cdots = q_N$), Dykeman, Ammar, and Wong [3] conjectured that the *Most Customers First* (MCF) policy that selects the page with the largest number of pending requests minimizes the mean request response time. They applied Howard's policy-iteration algorithm [4] to check their conjecture for some special cases under Markovian assumptions.

In this paper we provide a unified model and show, in Section 3, that under fairly weak statistical assumptions (in particular, no Markovian assumptions are needed) a cyclic policy minimizes the mean number of customers in the D-MQSS system. In Section 4 we show that the conjecture of Dykeman, Ammar, and Wong for the C-MQSS model actually holds true in a more general framework. More precisely, we prove that the MCF policy minimizes, in the sense of *strong stochastic ordering*, the vector of the number of customers in each queue whose components are arranged in decreasing order. These optimality results hold for both variants I and II.

2 Notation and Definitions

In this section, we introduce some notation and definitions that will be used throughout the paper. All the random variables considered in this paper are defined on some fixed probability triple $(\Omega, \mathcal{F}, \mathrm{P})$. The corresponding expectation operator is denoted by E.

We now describe the model under consideration. Customers arrive at random times $\{a_n\}_1^\infty$, with $0 < a_1 < a_2 < \cdots < a_n < \cdots$. At time a_n, the n-th customer joins queue $i_n \in \{1, 2, \ldots, N\}$

with probability $1/N$, i.e.,

$$P(i_n = j) = \frac{1}{N}, \tag{2.1}$$

for all $n \geq 1$, $j = 1, 2, \ldots, N$.

The n-th visit time of the server is modeled as a Random Variable (RV) $\sigma_n > 0$ a.s., $n \geq 1$. The sequences $\{a_n\}_1^\infty$, $\{\sigma_n\}_1^\infty$ and $\{i_n\}_1^\infty$ satisfy the condition

(H) $\{i_n\}_1^\infty$ is independent of $\{a_n, \sigma_n\}_1^\infty$,

but are otherwise arbitrary sequences. Any realization of sequences $\{a_n\}_1^\infty$ and $\{\sigma_n\}_1^\infty$ will be denoted by A and S, respectively. In particular, if $A = \{\alpha_n\}_1^\infty$ and $S = \{s_n\}_1^\infty$ we employ the shorthand

$$E_{A,S}(\bullet) := E(\bullet \mid a_1 = \alpha_1, a_2 = \alpha_2, \ldots; \sigma_1 = s_1, \sigma_2 = s_2, \ldots).$$

We also define b_n to be the time at which the n-th visit begins. We point out that under the above assumptions concerning the server's allocation, we always have $b_n + \sigma_n = b_{n+1}$ for the D-MQSS systems (both variants I and II), whereas $b_n + \sigma_n \leq b_{n+1}$ for the C-MQSS systems, where strict inequality holds if and only if the system is empty at time $b_n + \sigma_n$.

At the n-th decision epoch, an *admissible* policy π generates a (possibly randomized) action $\pi_n \in \{1, 2, \ldots, N\}$ on the basis of the available information $\mathbb{H}_n$, $n \geq 1$. We shall say that π_n is the n-th decision made by policy π, or equivalently stated, π_n denotes the n-th queue that is visited. The information sets $\{\mathbb{H}_n\}_1^\infty$ associated with the D-MQSS and with the C-MQSS models will be introduced in Section 3 and Section 4, respectively.

For $1 \leq m < \infty$ and $m \leq n \leq \infty$, denote by $\pi_{m,n}$ the set of decisions that policy π makes between the m-th and the n-th decision epochs (including those epochs), with the convention that $\pi_{n,n} := \pi_n$.

Finally, let $\mathbf{Q}_t(\pi) := \left(Q_t^1(\pi), \ldots, Q_t^N(\pi)\right)$, $t > 0$, where $Q_t^j(\pi)$ is the number of customers in queue j at time t under policy π, and define $\mathbf{Q}_0 := \left(Q_0^1, \ldots, Q_0^N\right)$ to be the state of the system at time 0.

3 Optimality of Cyclic Policy in the Decentralized Multi-Queue Single-Server Model

In this section, we consider the optimal scheduling problem in the decentralized multi-queue single-server system. We first discuss the variant of the D-MQSS model where the server only processes customers that are present in the queue upon its arrival (variant II).

We want to find a policy that minimizes

$$J_{t,x}(\pi) := \sum_{i=1}^{N} \mathrm{E}\left[Q_t^i(\pi)\right], \tag{3.1}$$

the expected number of customers in the system at time $t \geq 0$, where

$$x := Q_0^1 = Q_0^2 = \cdots = Q_0^N, \tag{3.2}$$

i.e., all queues start with the same number of customers. We assume in this section that x is fixed in $\mathbb{N}$, where $\mathbb{N} := \{0, 1, 2, \ldots\}$.

Additional notation and definitions are now introduced.

Statistical Assumptions. We assume that (2.1) and (H) are satisfied. Note that assumption (H) trivially holds if $\{i_n\}_1^\infty$, $\{a_n\}_1^\infty$ and $\{\sigma_n\}_1^\infty$ are mutually independent sequences.

Admissible Policy. For the D-MQSS model, a policy π is a deterministic sequence $\{\pi_n\}_1^\infty$, where $\pi_n \in \{1, 2, \ldots, N\}$ indicates the index of the n-th queue to be visited by the server, $n \geq 1$. Formally, this means that the set $\mathbb{H}_n$ mentioned in Section 2 simply contains the control actions prior to the n-th decision epoch, i.e., $\mathbb{H}_n = \{\pi_1, \pi_2 \ldots, \pi_{n-1}\}$ for $n \geq 2$, and $\mathbb{H}_n = \emptyset$ for $n = 1$.

We assume that $b_1 = 0$ and we recall that $b_{n+1} = b_n + \sigma_n$, $n \geq 1$ (cf. Section 2). The following assumption will avoid heavy notation: assume that at time b_n, $n \geq 1$, the server is allocated to a queue that has not yet been visited. Then, we assume without loss of generality because of assumptions (2.1) and (3.2), that

$$\pi_n = \text{the smallest queue index among the queues that have not yet been visited.} \tag{3.3}$$

Any deterministic policy satisfying (3.3) is called an *admissible* policy. In particular, $\pi_1 = 1$ for all admissible policies.

Cyclic Policy. For any admissible policy π, define

$$I_n^j(\pi) = \begin{cases} 0, & \text{if } \pi_i \neq j \text{ for all } i = 1, 2, \ldots, n-1; \\ \max\{1 \leq i \leq n-1 \, : \, \pi_i = j\}, & \text{otherwise,} \end{cases}$$

for $n \geq 2$, and $I_1^j(\pi) = 0$ for $1 \leq j \leq N$. $I_n^j(\pi)$ is the most recent (decision) epoch at which the server visited queue j. Note that $I_2^1(\pi) = 1$ and $I_2^j(\pi) = 0$ for $2 \leq j \leq N$ because of (3.3).

Fix $n \geq 1$ and let $\pi = \{\pi_m\}_1^\infty$ be an arbitrary admissible policy. The policy π is called a *cyclic policy of order* n if and only if for all $m \geq n$, $j = 1, 2, \ldots, N$,

$$\pi_m = j \Longrightarrow \forall k \in \{1, 2, \ldots, N\} - \{j\}, \quad I_m^j(\pi) \leq I_m^k(\pi). \tag{3.4}$$

In other words, if π is a cyclic policy of order n then $\pi_m = j$ implies that all other queues were visited more recently than queue j, for all $m \geq n$.

A cyclic policy of order 1 is called a *cyclic policy* and denoted by π°. Note that necessarily $\pi^\circ = (1, 2, \ldots, N, 1, 2, \ldots, N, \ldots)$ from assumption (3.3).

Finally, let $n \geq 1$ and let π be any admissible policy. We denote by $\pi^{(n)}$ the *unique* cyclic policy of order n that follows π in the first n steps (the uniqueness of $\pi^{(n)}$ is ensured by assumption (3.3)).

To illustrate the definition of $\pi^{(n)}$ consider the following example: take $N = 5$ and assume that $(1, 2, 1, 3)$ are the first decisions made by policy π. Then, cf. (3.3),

$$\begin{aligned} \pi_{1,4}^{(5)} &= (1, 2, 1, 3); \\ \pi_{5,\infty}^{(5)} &= (4, 5, 2, 1, 3, 4, 5, 2, 1, 3, \ldots). \end{aligned} \tag{3.5}$$

We now state the main result of this section.

Theorem 3.1 *The cyclic policy π° is optimal: For all $x \geq 0$, $t \geq 0$ and for any admissible policy π,*

$$\sum_{i=1}^N E\left[Q_t^i(\pi^\circ)\right] \leq \sum_{i=1}^N E\left[Q_t^i(\pi)\right].$$

In order to establish this result, we need the following lemma.

Lemma 3.1 *Let π be an arbitrary admissible policy. Then for all $t \geq 0$, $n \geq 1$, $x \geq 0$,*

$$\sum_{i=1}^N E_{A,S}\left[Q_t^i\left(\pi^{(n)}\right)\right] \leq \sum_{i=1}^N E_{A,S}\left[Q_t^i\left(\pi^{(n+1)}\right)\right].$$

Proof of the lemma Let π be an arbitrary admissible policy. Fix $n \geq 1$ and let $\pi_n = k$ for some $1 \leq k \leq N$. Assume that both sequences A and S are given. Let $j_1, j_2, \ldots, j_N$ be the queue indices such that $\forall i \in \{1, 2, \ldots, N\}$, either

$$I_n^{j_i}(\pi) < I_n^{j_{i+1}}(\pi),$$

or

$$I_n^{j_i}(\pi) = I_n^{j_{i+1}}(\pi) = 0 \quad \text{and} \quad j_i < j_{i+1},$$

according to our convention (3.3) (in (3.5), $I_5^1(\pi) = 3$, $I_5^2(\pi) = 2$, $I_5^3(\pi) = 4$, $I_5^4(\pi) = 0$, $I_5^5(\pi) = 0$; therefore, $j_1 = 4$, $j_2 = 5$, $j_3 = 2$, $j_4 = 1$ and $j_5 = 3$). Let $k = j_q$, $1 \leq q \leq N$.

From the definition of policy $\pi^{(k)}$, $k \geq 1$, it is readily seen that

$$\pi_m^{(n)} = \pi_m^{(n+1)} = \pi_m, \qquad \text{for } 1 \leq m \leq n-1, \tag{3.6}$$

and

$$\begin{aligned} \pi_{n,\infty}^{(n)} &= (j_1, j_2, \ldots, j_{q-1}, k, j_{q+1}, \ldots, j_N, j_1, j_2, \ldots, j_{q-1}, k, j_{q+1}, \ldots, j_N, \ldots); && (3.7) \\ \pi_{n,\infty}^{(n+1)} &= (k, j_1, \ldots, j_{q-2}, j_{q-1}, j_{q+1} \ldots, j_N, k, j_1, \ldots, j_{q-2}, j_{q-1}, j_{q+1} \ldots, j_N, \ldots). && (3.8) \end{aligned}$$

From (3.6), we immediately deduce that

$$\sum_{i=1}^{N} \mathrm{E}_{A,S}\left[Q_t^i\left(\pi^{(n)}\right)\right] = \sum_{i=1}^{N} \mathrm{E}_{A,S}\left[Q_t^i\left(\pi^{(n+1)}\right)\right], \tag{3.9}$$

for all $t \in [0, b_{n+1})$.

Let us now show that

$$\sum_{i=1}^{N} \mathrm{E}_{A,S}\left[Q_t^i\left(\pi^{(n)}\right)\right] \leq \sum_{i=1}^{N} \mathrm{E}_{A,S}\left[Q_t^i\left(\pi^{(n+1)}\right)\right], \tag{3.10}$$

for all $t \geq b_{n+1}$, which will conclude the proof.

In order to prove (3.10), define $M_{A,S}^j(a,b)$ to be the expected number of customers that join queue j in the time interval (a,b), given the sequences A and S, $j = 1, 2, \ldots, N$. Let us prove that $M_{A,S}^j(a,b)$ does not depend on j. We have:

$$M_{A,S}^j(a,b) \quad = \quad \mathrm{E}_{A,S}\left[\sum_{n \geq 1} 1\left(a_n \in (a,b), i_n = j\right)\right],$$

$$= \sum_{n=1}^{\infty} \mathbf{1}(a_n \in (a,b)) \mathrm{E}_{A,S} \left[\mathbf{1}(i_n = j) \right],$$

$$= \sum_{n=1}^{\infty} \mathbf{1}(a_n \in (a,b)) \mathrm{E} \left[\mathbf{1}(i_n = j) \right], \tag{3.11}$$

where the last equality follows from the fact that $\{i_n\}_1^\infty$ is independent of $\{a_n, \sigma_n\}_1^\infty$. Now, using (2.1) and (3.11), we get

$$M^j_{A,S}(a,b) = \frac{1}{N} \sum_{n=1}^{\infty} \mathbf{1}(a_n \in (a,b)), \tag{3.12}$$

which shows that $M^j_{A,S}(a,b)$ does not depend on j. In the following, the superscript j will be omitted.

Define now

$$\delta^j_t(\pi) = \begin{cases} \max\{b_n < t \,:\, \pi_n = j\}, & \text{if queue } j \text{ is visited in } [0,t); \\ 0, & \text{otherwise.} \end{cases}$$

If queue j is visited in $[0,t)$, then $\delta^j_t(\pi)$ is a RV that simply gives the beginning time of the last visit of the server to queue j.

We next show that

$$\sum_{i=1}^{N} \mathrm{E}_{A,S} \left[Q^i_{b_{n+l}} \left(\pi^{(n+1)} \right) \right] - \sum_{i=1}^{N} \mathrm{E}_{A,S} \left[Q^i_{b_{n+l}} \left(\pi^{(n)} \right) \right] = M_{A,S} \left(\delta^{j_l}_{b_n}(\pi), \delta^k_{b_n}(\pi) \right), \tag{3.13}$$

for $1 \le l \le q-1$, and

$$\sum_{i=1}^{N} \mathrm{E}_{A,S} \left[Q^i_{b_{n+l}} \left(\pi^{(n+1)} \right) \right] - \sum_{i=1}^{N} \mathrm{E}_{A,S} \left[Q^i_{b_{n+l}} \left(\pi^{(n)} \right) \right] = 0, \tag{3.14}$$

for $l \ge q$.

Let us first establish (3.13). For $l = 1$ we have, cf. (3.7)–(3.9) and (3.12),

$$\begin{aligned} &\sum_{i=1}^{N} \mathrm{E}_{A,S} \left[Q^i_{b_{n+1}} \left(\pi^{(n+1)} \right) \right] - \sum_{i=1}^{N} \mathrm{E}_{A,S} \left[Q^i_{b_{n+1}} \left(\pi^{(n)} \right) \right] \\ &= \mathrm{E}_{A,S} \left[Q^{j_1}_{b_n} \left(\pi^{(n)} \right) - Q^k_{b_n} \left(\pi^{(n+1)} \right) \right], \\ &= M_{A,S} \left(\delta^{j_1}_{b_n}(\pi), b_n \right) - M_{A,S} \left(\delta^k_{b_n}(\pi), b_n \right), \qquad (3.15) \\ &= M_{A,S} \left(\delta^{j_1}_{b_n}(\pi), \delta^k_{b_n}(\pi) \right). \end{aligned}$$

Assume that (3.13) holds for $1 \leq l \leq m-1$ with $m < q-1$. Then,

$$
\begin{aligned}
&\sum_{i=1}^{N} \mathrm{E}_{A,S}\left[Q^i_{b_{n+m}}\left(\pi^{(n+1)}\right)\right] - \sum_{i=1}^{N} \mathrm{E}_{A,S}\left[Q^i_{b_{n+m}}\left(\pi^{(n)}\right)\right] \\
&= \sum_{i=1}^{N} \mathrm{E}_{A,S}\left[Q^i_{b_{n+m-1}}\left(\pi^{(n+1)}\right)\right] - \sum_{i=1}^{N} \mathrm{E}_{A,S}\left[Q^i_{b_{n+m-1}}\left(\pi^{(n)}\right)\right] \\
&\quad - \mathrm{E}_{A,S}\left[Q^{j_{m-1}}_{b_{n+m-1}}\left(\pi^{(n+1)}\right) - Q^{j_m}_{b_{n+m-1}}\left(\pi^{(n)}\right)\right], \\
&= M_{A,S}\left(\delta^{j_{m-1}}_{b_n}(\pi), \delta^k_{b_n}(\pi)\right) - \mathrm{E}_{A,S}\left[Q^{j_{m-1}}_{b_{n+m-1}}\left(\pi^{(n+1)}\right) - Q^{j_m}_{b_{n+m-1}}\left(\pi^{(n)}\right)\right], \text{(induction hypothesis)} \\
&= M_{A,S}\left(\delta^{j_{m-1}}_{b_n}(\pi), \delta^k_{b_n}(\pi)\right) - M_{A,S}\left(\delta^{j_{m-1}}_{b_n}(\pi), \delta^{j_m}_{b_n}(\pi)\right), \qquad (3.16) \\
&= M_{A,S}\left(\delta^{j_m}_{b_n}(\pi), \delta^k_{b_n}(\pi)\right).
\end{aligned}
$$

Therefore, by induction, (3.13) holds for all $1 \leq l \leq q-1$.

Let us now proceed with the proof of (3.14). We have

$$
\begin{aligned}
&\sum_{i=1}^{N} \mathrm{E}_{A,S}\left[Q^i_{b_{n+q}}\left(\pi^{(n+1)}\right)\right] - \sum_{i=1}^{N} \mathrm{E}_{A,S}\left[Q^i_{b_{n+q}}\left(\pi^{(n)}\right)\right] \\
&= \sum_{i=1}^{N} \mathrm{E}_{A,S}\left[Q^i_{b_{n+q-1}}\left(\pi^{(n+1)}\right)\right] - \sum_{i=1}^{N} \mathrm{E}_{A,S}\left[Q^i_{b_{n+q-1}}\left(\pi^{(n)}\right)\right] \\
&\quad - \mathrm{E}_{A,S}\left[Q^{j_{q-1}}_{b_{n+q-1}}\left(\pi^{(n+1)}\right) - Q^k_{b_{n+q-1}}\left(\pi^{(n)}\right)\right], \\
&= M_{A,S}\left(\delta^{j_{q-1}}_{b_n}(\pi), \delta^k_{b_n}(\pi)\right) - M_{A,S}\left(\delta^{j_{q-1}}_{b_n}(\pi), \delta^k_{b_n}(\pi)\right), \qquad (3.17) \\
&= 0.
\end{aligned}
$$

Assume that (3.14) holds for $l = q, \ldots, m-1$. Then,

$$
\begin{aligned}
&\sum_{i=1}^{N} \mathrm{E}_{A,S}\left[Q^i_{b_{n+m}}\left(\pi^{(n+1)}\right)\right] - \sum_{i=1}^{N} \mathrm{E}_{A,S}\left[Q^i_{b_{n+m}}\left(\pi^{(n)}\right)\right] \\
&= \mathrm{E}_{A,S}\left[Q^{\pi^{(n)}_{n+m-1}}_{b_{n+m-1}}\left(\pi^{(n)}\right) - Q^{\pi^{(n+1)}_{n+m-1}}_{b_{n+m-1}}\left(\pi^{(n+1)}\right)\right], \\
&= 0,
\end{aligned}
$$

where we have used the properties (cf. (3.7), (3.8)) that for $q+1 \leq l \leq N$,

$$\pi^{(n)}_{n+l-1} = \pi^{(n+1)}_{n+l-1},$$

and that for $l > N$,

$$\delta_{b_n}^{\pi^{(n)}_{n+l-1}}\left(\pi^{(n)}\right) = \delta_{b_n}^{\pi^{(n+1)}_{n+l-1}}\left(\pi^{(n+1)}\right).$$

Again, by induction, (3.14) holds for $l \geq q$.

Combining (3.13) and (3.14) we get (3.10), which completes the proof. ∎

Proof of Theorem 3.1 Fix $t > 0$. Let π be an arbitrary admissible policy. From Lemma 3.1, we obtain

$$\sum_{i=1}^{N} \mathrm{E}_{A,S}\left[Q_t^i(\pi^\circ)\right] \leq \sum_{i=1}^{N} \mathrm{E}_{A,S}\left[Q_t^i\left(\pi^{(n)}\right)\right], \tag{3.18}$$

for all $n \geq 1$.

Now, by the construction of $\pi^{(n)}$ we know that there exists a finite integer $n_{A,S}$ such that for all $n \geq n_{A,S}$

$$\sum_{i=1}^{N} \mathrm{E}_{A,S}\left[Q_t^i\left(\pi^{(n)}\right)\right] = \sum_{i=1}^{N} \mathrm{E}_{A,S}\left[Q_t^i(\pi)\right]. \tag{3.19}$$

Combining (3.18) and (3.19) yields

$$\sum_{i=1}^{N} \mathrm{E}_{A,S}\left[Q_t^i(\pi^\circ)\right] \leq \sum_{i=1}^{N} \mathrm{E}_{A,S}\left[Q_t^i(\pi)\right]. \tag{3.20}$$

The proof is now completed by removing the conditioning on A and S in (3.20). ∎

The variant of the D-MQSS model when the server empties the visited queue at departure epochs can be analyzed in a similar way and the same result can be obtained (variant I). The only difference appears in the right-hand side of (3.13) that must now be equal to the mean number of customers that join any queue in the time interval $\left(\gamma_{b_n}^{j_l}(\pi), \gamma_{b_n}^{k}(\pi)\right)$, $1 \leq l \leq q-1$, where

$$\gamma_t^j(\pi) = \begin{cases} \max\{d_n < t \, : \, \pi_n = j\}, & \text{if queue } j \text{ is visited in } [0,t); \\ 0, & \text{otherwise,} \end{cases}$$

for $t > \sigma_1$, $1 \leq j \leq N$.

Remark 3.1 In the proof of (3.13) and (3.14), assumption (3.2) is used to ensure that, cf. (3.16), (3.17),

$$\mathrm{E}_{A,S}\left[Q_{b_{n+m-1}}^{j_{m-1}}\left(\pi^{(n+1)}\right) - Q_{b_{n+m-1}}^{j_m}\left(\pi^{(n)}\right)\right] \geq 0,$$

when $\delta_{b_n}^{j_{m-1}}(\pi) = \delta_{b_n}^{j_m}(\pi) = 0$, for $2 \le l \le q$, and that, cf. (3.15),

$$\mathrm{E}_{A,S}\left[Q_{b_n}^{j_1}\left(\pi^{(n)}\right) - Q_{b_n}^{k}\left(\pi^{(n+1)}\right)\right] \ge 0,$$

when $\delta_{b_n}^{j_1}(\pi) = \delta_{b_n}^{k}(\pi) = 0$.

4 Optimality of the MCF Policy in the Centralized Multi-Queue Single-Server Model

In this section we address the problem of determining an optimal scheduling policy for the centralized multi-queue single-server system. The discussion will first focus on the variant of the C-MQSS model where the server only processes the customers that are present in the queue upon its arrival (variant II). The objective is to find a policy that minimizes

$$G_{t,\mathbf{x}}(\pi) := \mathrm{E}\left[f\left(\mathcal{R}\mathbf{Q}_t(\pi)\right)\right], \tag{4.1}$$

for all $t \ge 0$, $\mathbf{Q}_0 = \mathbf{x} \in \mathbb{N}^{\mathrm{N}}$ and any nondecreasing mapping $f : \mathbb{N}^{\mathrm{N}} \to \mathbb{N}$, where $\mathcal{R} : \mathbb{N}^{\mathrm{N}} \to \mathbb{N}^{\mathrm{N}}$ is the nonincreasing permutation operator (for instance, if $N = 2$ then $\mathcal{R}(x_1, x_2) = (\max(x_1, x_2), \min(x_1, x_2))$).

Alternatively stated, a policy is optimal if it minimizes the N-dimensional vector $\mathcal{R}\mathbf{Q}_t(\pi)$ in the sense of *strong stochastic ordering* (denoted as usual by "$\le_{st}$"), for all $t \ge 0$, $\mathbf{x} \in \mathbb{N}^{\mathrm{N}}$ (see [8], pp. 256–257).

From now on, $\mathbf{x}$ is an arbitrary vector fixed in $\mathbb{N}^{\mathrm{N}}$. We introduce the assumptions and notation that will be needed for the analysis.

Statistical Assumptions. We assume that both (2.1) and condition (H) are satisfied. We further assume that

$$\mathrm{P}\left(i_{n_1} = x_1, \ldots, i_{n_p} = x_p\right) = \mathrm{P}\left(i_{n_1} = y_1, \ldots, i_{n_p} = y_p\right), \tag{4.2}$$

for all $x_i, y_i \in \{1, 2, \ldots, N\}$, $p \ge 1$, $n_1 < n_2 < \cdots < n_p$, $i = 1, 2, \ldots, p$.

Note that (4.2) necessarily implies that the i_n's are uniformly distributed and that (4.2) is satisfied if the i_n's are i.i.d. RV's.

Admissible Policy. For the C-MQSS model, the assignment of the n queue to be visited $n \ge 1$, will be based on the knowledge of past and present queue length vectors $\mathbf{x}, \mathbf{Q}_{b_1}(\pi), \ldots, \mathbf{Q}_{b_n}(\pi)$, and past control values. Consequently (cf. Section 2),

$$\mathbb{H}_n = \{\mathbf{x}, \mathbf{Q}_{b_1}(\pi), \ldots, \mathbf{Q}_{b_n}(\pi), \pi_1, \pi_2, \ldots, \pi_{n-1}\},$$

for $n \geq 2$, and

$$\mathbb{H}_1 = \{\mathbf{x}, \mathbf{Q}_{b_1}(\pi)\}.$$

We shall restrict ourselves to *nonpreemptive* policies as well as to policies that keep the server idles *if and only if* the system is empty. This implies that only nonidling or work conserving policies will be considered. In the following, an admissible policy will be any policy that satisfies the above conditions.

The *Most Customers First* (MCF) rule will denote the policy in which the server always visits the queue with the largest number of customers and breaks ties in an arbitrary manner.

The MCF policy will be denoted by the symbol π^{MCF}. In accordance with the previous notation, $\pi_{1,n}^{MCF}\pi_{n+1,\infty}$ will denote a policy that follows the MCF rule in the first n steps, and then switches to some policy π afterwards.

We start with the following lemma.

Lemma 4.1 *Fix $n \geq 1$ and p, $q \in \{1,2,\ldots,N\}$. From the sequences $\{a_m\}_1^\infty$, $\{\sigma_m\}_1^\infty$, $\{i_m\}_1^\infty$, we generate a new switching sequence $\{i'_m\}_1^\infty$ as follows:*

$$i'_m = i_m, \text{ for all } m \text{ such that } a_m \leq b_n; \tag{4.3}$$

$$i'_m = i_m \mathbf{1}(i_m \neq p, i_m \neq q) + p\,\mathbf{1}(i_m = q) + q\,\mathbf{1}(i_m = p), \text{ for all } m \text{ such that } a_m > b_n. \tag{4.4}$$

Then, $\{i_m\}_1^\infty$ and $\{i'_m\}_1^\infty$ are identical in law, and moreover, $\{i'_m\}_1^\infty$ is independent of $\{a_m, \sigma_m\}_1^\infty$.

Proof Let us first show that the sequences $\{i_m\}_1^\infty$ and $\{i'_m\}_1^\infty$ are identical in law. Consider the case where $x_1 = p$, $x_2 \neq p$, $x_2 \neq q$, $n_1 < n_2$. We have, cf. (4.3), (4.4),

$$\begin{aligned}
&\mathrm{P}\left(i'_{n_1} = x_1, i'_{n_2} = x_2\right) \\
&= \mathrm{P}\left(i'_{n_1} = x_1, i'_{n_2} = x_2, b_n < a_{n_1}\right) + \mathrm{P}\left(i'_{n_1} = x_1, i'_{n_2} = x_2, a_{n_1} \leq b_n < a_{n_2}\right) \\
&\quad + \mathrm{P}\left(i'_{n_1} = x_1, i'_{n_2} = x_2, a_{n_2} \leq b_n\right), \\
&= \mathrm{P}\left(i_{n_1} = q, i_{n_2} = x_2, b_n < a_{n_1}\right) + \mathrm{P}\left(i_{n_1} = x_1, i_{n_2} = x_2, a_{n_1} \leq b_n < a_{n_2}\right) \\
&\quad + \mathrm{P}\left(i_{n_1} = x_1, i_{n_2} = x_2, a_{n_2} \leq b_n\right), \\
&= \mathrm{P}\left(i_{n_1} = x_1, i_{n_2} = x_2\right),
\end{aligned}$$

where we used the independence assumption between $\{i_m\}_1^\infty$ and $\{a_m; \sigma_m\}_1^\infty$, together with (4.2). The general proof is similar and is omitted for sake of conciseness.

It remains to establish that $\{i'_m\}_1^\infty$ is independent of $\{a_m, \sigma_m\}_1^\infty$. For $x_1 = p$, $x_2 \neq p$, $x_2 \neq q$, $n_1 < n_2$, it is readily seen that, cf. (4.3), (4.4),

$$\begin{aligned}
& \mathrm{P}\left(i'_{n_1} = x_1, i'_{n_2} = x_2 \mid A, S\right) \\
= \;& \mathrm{P}\left(i_{n_1} = q, i_{n_2} = x_2, b_n < a_{n_1} \mid A, S\right) + \mathrm{P}\left(i_{n_1} = x_1, i_{n_2} = x_2, b_n \geq a_{n_1} \mid A, S\right), \\
= \;& \mathrm{P}\left(i_{n_1} = q, i_{n_2} = x_2, \mid b_n < a_{n_1}, A, S\right)\mathrm{P}\left(b_n < a_{n_1} \mid A, S\right) \\
& + \mathrm{P}\left(i_{n_1} = x_1, i_{n_2} = x_2 \mid b_n \geq a_{n_1}, A, S\right)\mathrm{P}\left(b_n \geq a_{n_1} \mid A, S\right), \\
= \;& \mathrm{P}\left(i_{n_1} = x_1, i_{n_2} = x_2\right),
\end{aligned} \tag{4.5}$$

where we used (4.2) and the fact that $\{i_m\}_1^\infty$ is independent of $\{a_m, \sigma_m\}_1^\infty$. Next, using the property that $\{i_m\}_1^\infty$ and $\{i'_m\}_1^\infty$ are identical in law, we get from (4.5),

$$\mathrm{P}\left(i'_{n_1} = x_1, i'_{n_2} = x_2 \mid A, S\right) = \mathrm{P}\left(i'_{n_1} = x_1, i'_{n_2} = x_2\right).$$

Again, the general proof is omitted for sake of conciseness. ∎

Lemma 4.2 *Let π be an admissible policy such that $\pi = \pi_{1,n-1}^{MCF}\,\pi_{n,\infty}$, for some $n \geq 1$. Then, there exists a policy $\pi' := \pi_{1,n}^{MCF}\,\pi_{n+1,\infty}^{[n]}$ such that*

$$E_{A,S}\left[f\left(\mathcal{R}\mathbf{Q}_t(\pi')\right)\right] \leq E_{A,S}\left[f\left(\mathcal{R}\mathbf{Q}_t(\pi)\right)\right], \tag{4.6}$$

for all $t \geq 0$, $\mathbf{x} \in \mathbb{N}^N$, and for any nondecreasing mapping $f : \mathbb{N}^N \to \mathbb{N}$ such that the expectations in (4.6) exist.

Proof Assume that the sequences A and S are fixed. Let $p = \pi_n$ and let q be the queue index such that

$$Q_{b_n}^q(\pi) = \max_{1 \leq i \leq N} Q_{b_n}^i(\pi).$$

In other words, q is the index of the largest queue at time b_n, under policy π.

We construct a new policy π' as follows:

$$\pi'_{1,n} \;:=\; \pi_{1,n}^{MCF}; \tag{4.7}$$

$$\pi'_m := \pi_m^{[n]} \;:=\; \pi_m\,\mathbf{1}\left(\pi_m \neq p, \pi_m \neq q\right) + p\,\mathbf{1}\left(\pi_m = q\right) + q\,\mathbf{1}\left(\pi_m = p\right), \tag{4.8}$$

for all $m \geq n+1$. In words, the policy π' follows π, except that it serves queue p (resp. queue q) when π serves queue q (resp. p) after the first n steps.

The policy π' operates on the sequences $\{i'_n\}_1^\infty$, A and S, where i'_n was defined in Lemma 4.1, whereas the policy π operates on the sequences $\{i_n\}_1^\infty$, A and S. The following results hold:

$$Q_t^i(\pi') = Q_t^i(\pi), \text{ for } 0 \leq t \leq b_n,\ 1 \leq i \leq N, \tag{4.9}$$

and

$$\begin{aligned} Q_t^i(\pi') &= Q_t^i(\pi), \text{ for } i \in \{1,2,\ldots,N\} - \{p,q\}; & (4.10)\\ Q_t^p(\pi') &\leq Q_t^q(\pi); & (4.11)\\ Q_t^q(\pi') &\leq Q_t^p(\pi), & (4.12)\end{aligned}$$

for all $t > b_n$.

Note first that (4.9) follows from the fact that policies π and π' are identical in $[0, b_n)$. Property (4.10) follows from the fact that under policies π and π' queue i has the same behavior for $i \neq p$, $i \neq q$ (see (4.3), (4.4), (4.7) and (4.8)).

By construction of π', the jump times of processes $\mathbf{P}^p(\pi) := \{Q_t^p(\pi), t > b_n\}$ and $\mathbf{P}^q(\pi') := \{Q_t^q(\pi'), t > b_n\}$ are the same, and similarly for p and q interchanged. It is worth noting, however, that the magnitude of simultaneous downward jumps (departures) in both processes is not necessarily the same, whereas the magnitude of all upward jumps (arrivals) is always 1. Let $b_n = t_0 < t_1 < t_2 < \ldots < t_k < \cdots$ denote the consecutive jump epochs after b_n of the process resulting from the superposition of $\mathbf{P}^p(\pi)$ and $\mathbf{P}^q(\pi)$. Clearly, inequalities (4.11) and (4.12) need only be proved for these particular values of t.

We use an induction argument for proving (4.11) and (4.12). We know that (4.11) and (4.12) hold for $t = t_0$. Assume they hold for $t_0 \leq t \leq t_{k-1}$, and let us show they still hold for $t = t_k$.

From the above comments concerning the upward jumps, it is clear that (4.11) and (4.12) hold at time t_k if this time corresponds to an arrival in either $\mathbf{P}^p(\pi)$ or $\mathbf{P}^q(\pi)$.

Consider now the case where t_k is a downward jump epoch, and assume, without loss of generality, that t_k is a jump epoch of $\mathbf{P}^p(\pi)$. Hence, t_k is a downward jump point of $\mathbf{P}^q(\pi')$. If the N queues are empty in both systems at time t_k, then (4.11) and (4.12) trivially hold at time t_k. If not, there exists an $r \geq n$ such that

$$b_{r-1} + \sigma_{r-1} = b_r,$$

with $b_r = t_k$, $\pi_{r-1} = p$ and $\pi'_{r-1} = q$.

Define $M^j(a,b)$ to be the number of customers that join queue j in the time interval (a,b), given A and S. From the very definition of π and π', we obtain

$$\begin{aligned} Q^p_{b_r}(\pi) &= M^p(b_{r-1}, b_r); & (4.13)\\ Q^q_{b_r}(\pi) &= Q^q_{b_{r-1}}(\pi) + M^q(b_{r-1}, b_r), & (4.14)\end{aligned}$$

and

$$\begin{aligned} Q^p_{b_r}(\pi') &= Q^p_{b_{r-1}}(\pi') + M^q(b_{r-1}, b_r); & (4.15)\\ Q^q_{b_r}(\pi') &= M^p(b_{r-1}, b_r). & (4.16)\end{aligned}$$

Since $Q^p_{b_{r-1}}(\pi') \leq Q^q_{b_{r-1}}(\pi)$ from the induction hypothesis, we get, from relations (4.13)–(4.16)

$$\begin{aligned} Q^p_{b_r}(\pi') &\leq Q^q_{b_r}(\pi);\\ Q^q_{b_r}(\pi') &= Q^p_{b_r}(\pi),\end{aligned}$$

which shows, by induction, that (4.11) and (4.12) are satisfied for all $t > b_n$.

Therefore, for all $t \geq 0$, cf. (4.9)–(4.12),

$$\mathcal{R}\mathbf{Q}_t(\pi') \leq \mathcal{R}\mathbf{Q}_t(\pi). \tag{4.17}$$

Owing to Lemma 4.1, it is seen that $\mathcal{R}\mathbf{Q}_t(\pi')$ has the same law under the input sequences $\{a_n, \sigma_n, i_n\}_1^\infty$ and $\{a_n, \sigma_n, i'_n\}_1^\infty$, which implies, using (4.17), that for all $t \geq 0$,

$$\mathcal{R}\mathbf{Q}_t(\pi') \leq_{st} \mathcal{R}\mathbf{Q}_t(\pi), \text{ given } A \text{ and } S,$$

or equivalently, that

$$\mathrm{E}_{A,S}\left[f\left(\mathcal{R}\mathbf{Q}_t(\pi')\right)\right] \leq \mathrm{E}_{A,S}\left[f\left(\mathcal{R}\mathbf{Q}_t(\pi)\right)\right], \tag{4.18}$$

for any nondecreasing mapping $f : \mathbb{N}^N \to \mathbb{N}$ such that the expectations in (4.18) exist. ∎

We are now ready to prove the main result of this section.

Theorem 4.1 *The MCF policy is optimal: For all* $\mathbf{x} \in \mathbb{N}^N$, $t \geq 0$ *and for any admissible policy* π,

$$\mathcal{R}\mathbf{Q}_t(\pi^{MCF}) \leq_{st} \mathcal{R}\mathbf{Q}_t(\pi).$$

Proof Let π be an arbitrary admissible policy such that $\pi = \pi^{MCF}_{1,n-1}\pi_{n,\infty}$ for some $1 \leq n < \infty$. If the MCF rule is not applied at time b_1, then π simply writes $\pi = \pi_{1,\infty}$. Applying Lemma 4.2. recursively with the same sequences A and S yields

$$\mathrm{E}_{A,S}\left[f\left(\mathcal{R}\mathbf{Q}_t\left(\pi^{MCF}_{1,l}\,\pi^{[l]}_{l+1,\infty}\right)\right)\right] \leq \mathrm{E}_{A,S}\left[f\left(\mathcal{R}\mathbf{Q}_t(\pi)\right)\right], \tag{4.19}$$

for all $t \geq 0$, $l \geq n$, and for all nondecreasing mappings $f : \mathbb{N}^N \to \mathbb{N}$ such that the expectations in (4.19) exist.

From inequality (4.19) we deduce that

$$\mathrm{E}_{A,S}\left[f\left(\mathcal{R}\mathbf{Q}_t\left(\pi^{MCF}\right)\right)\right] \leq \mathrm{E}_{A,S}\left[f\left(\mathcal{R}\mathbf{Q}_t(\pi)\right)\right], \tag{4.20}$$

for all $t \geq 0$ (see the proof of Theorem 3.1 where the same argument is used). The proof is completed by removing the conditioning on A and S in (4.20). ∎

The same result can be proved for the variant of the C-MQSS model where the server empties the visited queue (variant I). The only differences appear in the right-hand sides of equations (4.13) and (4.16), which must now be equal to 0 (cf. the proof of Lemma 4.2).

Remark 4.1 It is easily seen that the successive policies $\left\{\pi_{1,n}^{MCF}\,\pi_{n+1,\infty}^{[n]}\right\}_{n=1}^{\infty}$ generated in Lemma 4.2 are all admissible policies. However, Theorem 4.1 would still hold if these intermediate policies were not admissible (i.e., if they were anticipative) since the policy π^{MCF} shown to be optimal is an admissible policy (see [2] and [7] where this argument was used).

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